

18.100A PSET 4 SOLUTIONS

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PROBLEM 1

Let $f(x) = y^2 \cos x - e^x$. Then $f(0) = y^2 - 1$. As $y \geq 1$, we have $f(0) \geq 0$. As well, we have $f(\pi/2) = 0 - e^{\pi/2} = -e^{\pi/2} < 0$.

By the intermediate value theorem, since $f(0) \geq 0 \geq f(\pi/2)$, there is $x_0 \in [0, \pi/2]$ such that $f(x_0) = 0$. Since $f(\pi/2) \neq 0$, we know $x_0 \neq \pi/2$, so $x_0 \in [0, \pi/2)$.

Comments. While it might help conceptually to consider the cases $y = 1$ and $y > 1$ separately, you really don't need to do that. The intermediate value theorem applies equally well in each case.

PROBLEM 2

Suppose, for sake of contradiction, that $f(x)$ is not strictly increasing. This means that we can find $a \leq c < d \leq b$ such that $f(d) \leq f(c)$.

Since the range of f is $[f(a), f(b)]$, we know $f(a) \leq f(d) \leq f(c)$. It follows that there is $x_0 \in [a, c]$ such that $f(x_0) = f(d)$. But $x_0 \neq d$ because $x_0 \leq c < d$, which means that f repeats a value, a contradiction.

This contradiction implies that f is strictly increasing.

Comments. This was a lot simpler than many people made it out to be. You only need to use the intermediate value theorem once, and you really don't need to break it down into so many cases.

PROBLEM 3

(a). Let S_1, \dots, S_d be the intervals, and let S be their union.

If $\{x_n\}$ is a sequence contained in S , then there is some integer $i \in [1, d]$ such that S_i contains infinitely many elements of the sequence. We consider the subsequence of $\{x_n\}$ consisting of all $\{x_n\}$ such that $\{x_n\} \in S_i$. Then this subsequence is a sequence in S_i , so by compactness of S_i , it has a subsequence that converges to a limit in S_i (and hence in S).

In summary, any sequence $\{x_n\}$ in S has a convergent subsequence in S , so S is sequentially compact.

(b). Let $S_n = [n, n + 1]$ for integer n . Then the union of all the S_n 's is \mathbb{R} , which is not compact, even though each S_n is compact.

PROBLEM 4

(a). We let $f(x) = \frac{x}{1-x}$ for $x \in [0, 1)$ and $f(x) = 3352435$ for $x = 1$ (in fact, any positive number will do).

Then f is always positive, $f(0) = 0$, and $\lim_{x \rightarrow 1} f(x) = \infty$, so the intermediate value theorem tells us that f takes every positive real value. (More precisely, for any positive y , we can find c close to 1 such that $f(c) > y$, so the intermediate value theorem tells us that f takes the value y somewhere in the interval $[0, c]$.)

As an alternative, one could consider $f(x) = \tan(\pi x/2)$ for $x < 1$.

(b). A continuous function is bounded on a closed interval, so the range of f must be bounded. But $[0, \infty)$ is not bounded.

Comments. Some people considered $f(x) = 1/x$, but the problem with this function is that it does not hit the values of y that are between 0 and 1.

PROBLEM 5

(a). Suppose f has a minimum on I . Call that minimum m . Then since f is always positive, we have $m > 0$. But since $\lim_{x \rightarrow \infty} f(x) = 0$, we have $|f(x)| < m$ for x sufficiently large, a contradiction. This contradiction implies that there is no minimum.

(b,c). We do part (b) while only assuming that $f(x)$ takes at least one non-negative value on all of \mathbb{R} .

We divide into two cases.

In the first case, we assume that f does not take any positive values. Then by assumption, there is some c such that $f(c)$ is not negative, which implies that $f(c) = 0$. But then $f(c)$ is the maximum, so we are done in this case.

In the second case, we assume that f does take (at least one) positive value. Let c be such that $f(c) > 0$. As $\lim_{x \rightarrow \infty} f(x) = 0$, we can find $a > |c|$ such that $|f(x)| < f(c)$ whenever $|x| > a$. As f is continuous, it has a maximum value on the closed interval $[-a, a]$, call that value M . Then $f(c) \leq M$ because $c \in [-a, a]$, so for all $x \notin [-a, a]$ we have $f(x) \leq |f(x)| < f(c) \leq M$. As $f(x) \leq M$ for all $x \in [-a, a]$ as well, we find that M is the maximum of f .

Comments. As a much less weak assumption, one might assume that $f(x) \geq 0$ for all x . But in fact, the argument works even when assuming non-negativity for *only one* value of x .